TWELFTH-ORDER METHOD FOR NONLINEAR EQUATIONS

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ABSTRACT

Modification of Newton's method with higher-order convergence is presented. The modification of Newton's method is based on Bi's eighth-order method. Per iteration of the new method requires four-step. Analysis of convergence demonstrates that the order of convergence is 12. Some numerical examples illustrate that the algorithm is more efficient and performs better than classical Newton's method and other methods.

Keywords : *Nonlinear equations; Iterative methods; Newton's method; Order of convergence*

1 INTRODUCTION

In this paper, we apply iterative method to find a simple root x^* of the nonlinear equation $f(x)=0$, where $f: D \subset R \to R$ is a scalar function on an open interval D. It is well known that Newton's method is one of the best iterative methods for solving a single nonlinear equation by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which converges quadratically in some neighborhood of x^*. Many iterative methods have been developed by using many different techniques including quadrature formulas, Taylor series and decomposition techniques. For more details, see[1-20] and the references therein. King [2] developed a one-parameter family of fourth-order methods, which is written as:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - \dfrac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)} \end{cases} \tag{2}$$

Where $\beta \in R$ is a constant. In particular, the special method for $\beta = -0.5$ is as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - \dfrac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)} \dfrac{f(y_n)}{f'(x_n)} \end{cases} \tag{3}$$

Based on method (3), Bi et al. [17] developed a one-parameter family of eighth-order method, which is

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ z_n = y_n - \dfrac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)} \dfrac{f(y_n)}{f'(x_n)} \\ x_{n+1} = z_n - \dfrac{f(x_n)+(2+\alpha)f(y_n)}{f(x_n)+\alpha f(y_n)} \dfrac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)} \end{cases} \tag{4}$$

In this paper, based on method (4), we construct modification of Newton's method with higher-order convergence for solving nonlinear equations.

2 THE METHOD AND ANALYSIS OF CONVERGENCE

From (4), we construct four-step iterative method in this paper

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ z_n = y_n - \dfrac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\dfrac{f(y_n)}{f'(x_n)} \\ w_n = z_n - \dfrac{2f(x_n)-f(z_n)}{2f(x_n)-5(z_n)}\dfrac{f(z_n)}{F(x_n)} \\ x_{n+1} = w_n - \dfrac{f(x_n)+(2+\alpha)f(z_n)}{f(x_n)+\alpha f(z_n)}\dfrac{f(w_n)}{F(x_n)} \end{cases} \tag{5}$$

Where $F(x_n) = f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)$ (see[17]).

We state the following convergence theorem for the method (5).

Theorem 1 Suppose that $f \in C^2(D)$, assume that the function $f: D \subset R \to R$ has a single root $x^* \in D$, where D is an open interval. If the initial point x_0 is sufficiently close to x^*, then the method defined by (5) has twelfth-order convergence.

Proof: Let x^* be a single root of nonlinear equation $f(x)$. Using Taylor's expansion, we have

$$f(x_n) = f'(x^*)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + o(e_n^5)] \tag{6}$$

$$f'(x_n) = f'(x^*)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + o(e_n^4)] \tag{7}$$

$$f^{(2)}(x_n) = f'(x^*)[2c_2 + 6c_3e_n + 12c_4e_n^2 + o(e_n^3)] \tag{8}$$

$$f^{(3)}(x_n) = f'(x^*)[6c_3 + 24c_4e_n + o(e_n^2)] \tag{9}$$

$$f^{(4)}(x_n) = f'(x^*)[24c_4 + o(e_n)] \tag{10}$$

where $c_k = \dfrac{f^{(k)}(x*)}{k!\,f'(x*)}$, $k = 2,3...,$ and $e_n = x_n - x*$.

From (6) and (7), we obtain

$$d_n = -\frac{f(x_n)}{f'(x_n)} = -e_n + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + o(e_n^5) \tag{11}$$

Expanding

$f(x_n - \dfrac{f(x_n)}{f'(x_n)})$ about x_n, we get

$$f(x_n - \frac{f(x_n)}{f'(x_n)}) = f(y_n) = f(x_n) + f'(x_n)d_n + \frac{1}{2}f^{(2)}(x_n)d_n^2 + \frac{1}{3!}f^{(3)}(x_n)d_n^3 + \frac{1}{4!}f^{(4)}(x_n)d_n^4 + o(e_n^5) \tag{12}$$

Substituting (6)-(11) into (12), and simplifying, we have

$$f(x_n) = f'(x^*)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 + 5c_2^3 - 7c_2c_3)e_n^4 + o(e_n^5)] \tag{13}$$

With (6) and (13), using Taylor's expansion, and simplifying, we obtain

$$\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)} = 1+2c_2e_n+(4c_3-2c_2^2)e_n^2+(6c_4-\frac{3}{2}c_2^3)e_n^3 + (8c_5+2c_2c_4+\frac{15}{4}c_2^4-11c_3c_2^2+4c_3^2)e_n^4+o(e_n^5) \tag{14}$$

Dividing (13) by (7), using Taylor's expansion, and simplifying, we get

$$\frac{f(y_n)}{f'(x_n)} = c_2e_n^2+(2c_3-4c_2^2)e_n^3+(3c_4+13c_2^3-14c_2c_3)e_n^4+o(e_n^5) \tag{15}$$

Substituting (6), (7), (14), (15) into the second formula of (5), using Taylor's expansion, and simplifying, we have

$$z_n = x^* - c_2c_3e_n^4+(-2c_2c_4+2c_2^2c_3+\frac{3}{2}c_2^4-2c_3^2)e_n^5+o(e_n^6) \tag{16}$$

Similar to (13), we obtain

$$f(z_n) = f'(x^*)[-c_2c_3e_n^4+\frac{1}{2}(-4c_2c_4+4c_2^2c_3+3c_2^4-4c_3^2)e_n^5 + \frac{1}{4}(-12c_2c_5-28c_3c_4+12c_4c_2^2+24c_3c_2^3-35c_2^5)e_n^6+o(e_n^7)] \tag{17}$$

By (6),(16),(17), using Taylor's expansion, and simplifying, we get

$$\frac{f(z_n)-f(x_n)}{z_n-x_n} = f'(x^*)[1+c_2e_n+c_3e_n^2+c_4e_n^3+(c_5-c_3c_2^2)e_n^4 + c_2(-3c_3^2+\frac{3}{2}c_2^4-2c_2c_4+2c_3c_2^2)e_n^5+(3c_4c_2^3+8c_3^2c_2^2 + \frac{19}{2}c_3c_2^4-\frac{35}{4}c_2^6-2c_3^3-3c_2^2c_5-10c_2c_3c_4)e_n^6+o(e_n^7)] \tag{18}$$

With (11),(12),(16),(17), using Taylor's expansion, and simplifying, we obtain

$$\frac{f(z_n)-f(y_n)}{z_n-y_n} = f'(x^*)[1+c_2^2e_n^2+2c_2(c_3-c_2^2)e_n^3+c_2(3c_4-7c_2c_3+4c_2^3)e_n^4 + c_2(4c_5-12c_2c_4-4c_3^2+18c_3c_2^2-\frac{13}{2}c_2^4)e_n^5+(-16c_5c_2^2+32c_4c_2^3+16c_3^2c_2^2 - 32c_3c_2^4-18c_2c_3c_4+4c_3^3+\frac{29}{4}c_2^6)e_n^6+o(e_n^7)] \tag{19}$$

Substituting (7), (11), (16), (18), (19) into $F(x_n)$ of (5), using Taylor's expansion, and simplifying, we have

$$F(x_n) = \frac{f(z_n)-f(y_n)}{z_n-y_n} + \frac{\frac{f(z_n)-f(x_n)}{z_n-x_n}-f'(x_n)}{z_n-x_n}(z_n-y_n)$$
$$= f'(x^*)[1-2c_2c_3e_n^3+(3c_2^2c_3-3c_2c_4-4c_3^2)e_n^4+(-4c_2c_5-12c_3c_4+2c_2^2c_4 + 12c_2c_3^2-8c_3c_2^3+3c_2^5)e_n^5+o(e_n^6)] \tag{20}$$

Dividing (17) by (20), using Taylor's expansion, and simplifying, we get

$$\frac{f(z_n)}{F(x_n)} = -c_2c_3e_n^4+(-2c_3^2+\frac{3}{2}c_2^4-2c_2c_4+2c_3c_2^2)e_n^5 + (-3c_2c_5-7c_3c_4+3c_4c_2^2+6c_2c_3^2+8c_3c_2^3-\frac{35}{4}c_2^5)e_n^6+o(e_n^7) \tag{21}$$

From (6),(17), using Taylor's expansion, and simplifying, we obtain

$$\frac{2f(x_n)-f(z_n)}{2f(x_n)-5f(z_n)}=1-2c_2c_3e_n^3+(-4c_3^2+3c_2^4-4c_2c_4+6c_3c_2^2)e_n^4+(-6c_2c_5-14c_3c_4$$
$$+10c_4c_2^2+18c_2c_3^2+10c_3c_2^3-\frac{41}{2}c_2^5)e_n^5+(52c_2c_3c_4-148c_3c_2^4+\frac{337}{4}c_2^6 \qquad (22)$$
$$+12c_3^3-12c_4^2+18c_2^3c_4+13c_3^2c_2^2-20c_3c_5+14c_5c_2^2)e_n^6+o(e_n^7)$$

Substituting (16),(21),(22) into the third formula of (5), using Taylor's expansion, and simplifying, we have

$$w_n=z_n-\frac{2f(x_n)-f(z_n)}{2f(x_n)-5f(z_n)}\frac{f(z_n)}{F(x_n)}$$
$$=x^*+c_3c_2^2(3c_2^3+2c_2c_3-c_4)e_n^8-\frac{1}{2}c_2(4c_2c_3c_5+8c_4c_3^2-24c_2^2c_3c_4 \qquad (23)$$
$$+9c_2^7+62c_3c_2^5-16c_2c_3^3+4c_2c_4^2-15c_4c_2^4-44c_3^2c_2^3)e_n^9+o(e_n^{10})$$

Similar to (13), we obtain

$$f(w_n)=f'(x^*)[c_2^2c_3(3c_2^3+2c_2c_3-c_4)e_n^8-\frac{1}{2}c_2(4c_2c_3c_5+8c_4c_3^2-24c_2^2c_3c_4 \qquad (24)$$
$$+9c_2^7+62c_2^5c_3-16c_2c_3^3+4c_2c_4^2-15c_2^4c_4-44c_3^2c_2^3)e_n^9+o(e_n^{10})]$$

Similar to (22), we get

$$\frac{f(x_n)+(2+\alpha)f(z_n)}{f(x_n)+\alpha f(z_n)}=1-2c_2c_3e_n^3+(-4c_3^2+3c_2^4-4c_2c_4+6c_3c_2^2)e_n^4$$
$$+(-6c_2c_5-14c_3c_4+10c_4c_2^2+18c_2c_3^2+10c_3c_2^3-\frac{41}{2}c_2^5)e_n^5 \qquad (25)$$
$$+(52c_2c_3c_4-148c_3c_2^4+\frac{337}{4}c_2^6+12c_3^3-12c_4^2+18c_2^3c_4$$
$$+8c_3^2c_2^2-2c_2^2c_3^2\alpha-20c_3c_5+14c_5c_2^2)e_n^6+o(e_n^7)$$

Dividing (24) by (20), using Taylor's expansion, and simplifying, we obtain

$$\frac{f(w_n)}{F(x_n)}=c_3c_2^2(2c_2c_3-c_4+3c_2^3)e_n^8+c_2(12c_2^2c_3c_4-2c_2c_3c_5-4c_3^2c_4 \qquad (26)$$
$$-2c_2c_4^2+\frac{15}{2}c_2^4c_4+8c_2c_3^3+22c_2^3c_3^2-31c_2^5c_3-\frac{9}{2}c_2^7)e_n^9+o(e_n^{10})$$

Substituting (23),(25),(26) into the fourth formula of (5), using Taylor's expansion, and simplifying, we get the error equation:

$$x_{n+1}=w_n-\frac{f(x_n)+(2+\alpha)f(z_n)}{f(x_n)+\alpha f(z_n)}\frac{f(w_n)}{F(x_n)} \qquad (27)$$
$$=c_3c_2^3(5c_2c_3c_4+6c_4c_2^3-c_4^2-9c_2^6-6c_3^2c_2^2-15c_3c_2^4)e_n^{12}+o(e_n^{13})$$

This means that the method defined by (5) has twelfth-order convergence. The proof is completed.

Remark The order of convergence of the iterative method (5) is 12. Per iteration of the iterative method (5) requires four evaluations of the function, namely, $f(x_n), f(y_n), f(z_n), f(w_n)$ and one evaluation of first derivative $f'(x_n)$. We take into account the definition of efficiency index [21] as $p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we suppose that all the evaluations have the same cost as function one, we have that the efficiency index of the method (5) is $\sqrt[5]{12}\approx 1.644$ which is better than $\sqrt{2}\approx 1.414$ of Newton's method, $\sqrt[3]{3}\approx 1.442$ of method[8,19],

$\sqrt[3]{4} \approx 1.587$ of method[2], $\sqrt[3]{5} \approx 1.495$ of method[12], $\sqrt[4]{6} \approx 1.565$ of method[7], $\sqrt[4]{7} \approx 1.627$ of method[9,20].

3 NUMERICAL RESULTS

NC in Table 1 means that the method does not converge to the root. Now, we employ the four-step method(5)(G1) suggested in this paper to solve some nonlinear equations and compare them with three-step iterative methods with eighth-order convergence[17](G2, (36)), some variants of Ostrowskis method[9](G3,(8)), sixth-order variants of Ostrowski root-finding methods[7](G4,(28)-(30)), a family of fifth-order iterations composed of Newton methods[12](G5,(10)), a family of fourth order methods for nonlinear equations[2](G6,(2.4)), A modification of Newton method with third-order convergence[8](G7,(10)) and Newton's method(G8).

Table 1 Comparison of various iterative methods

F(x)	G1	G2	G2	G3	G4	G5	G6	G6	G7	G8
parameter	$\alpha=0$	$\alpha=2$	$\alpha=0$	$\alpha=2$	$\gamma=1$ $\beta=1$		$\beta=1$	$\beta=3$		
$f_1, x_0=4.9$	5	NC	NC	16	20	248	18	15	15	16
$f_2, x_0=-4.8$	10	12	12	16	20	20	18	NC	NC	NC
$f_3, x_0=-2.25$	10	12	12	16	20	NC	15	15	NC	12
$f_4, x_0=4.6$	10	12	12	16	20	20	24	18	18	20
$f_5, x_0=9.9$	10	12	12	12	20	NC	18	12	15	14
$f_6, x_0=1.5$	10	16	NC	12	24	NC	12	12	NC	NC
$f_7, x_0=8$	10	NC	NC	16	16	260	21	18	12	16
$f_8, x_0=4$	15	16	16	28	36	36	33	33	36	38
$f_9, x_0=9.8$	10	12	12	16	20	20	18	15	18	16
$f_{10}, x_0=15.5$	5	8	8	12	12	20	12	9	9	10
$f_{11}, x_0=7.7$	10	12	12	16	20	24	24	18	18	20
$f_{12}, x_0=11.9$	5	8	8	12	12	16	9	9	9	12

For example, it can be seen that the method(G2), the method(G5), the King's method(G6), the method(G7) and Newton's method(G8) have sensitivities to the original iteration value: the method(G2)[17] does not converge to the zero f_1 for $x_0=4.9$, f_6 for $x_0=1.5$ and f_7 for $x_0=8.0$, the method(G5)[12] does not converge to the zero f_3 for $x_0=-2.25$,

f_5 for $x_0=9.9$ and f_6 for $x_0=1.5$, the method(G6)[2] does not converge to the zero f_2 for $x_0=-1.8$, the method(G7)[8] does not converge to the zero f_2 for $x_0=-1.8$, f_3 for $x_0=-2.25$ and f_6 for $x_0=1.5$, Newton's method does not converge to the zero f_2 for $x_0=-1.8$, and f_6 for $x_0=1.5$. Displayed in Table 1 are the number of function evaluations (NFE) required such that $|f(x_n)|<1.E-17$, $|x_n-x^*|<1.E-17$. All computations were done by using Visual C++ 6.0. In Table 1 we use the following test functions and display the approximate zeros x^* found up to the 17th decimal place:

$$f_1(x)=(x-1)^3-1\ ,\ x^*=2.000000000000000.$$

$$f_2(x)=2x\cos(x)+x-3\ ,\ x^*=-3.03446643069740450.$$

$$f_3(x)=e^{-x^2+x+2}-x+2\ ,\ x^*=2.4905398276083051.$$

$$f_4(x)=e^x-1\ ,\ x^*=0.0000000000000000.$$

$f_5(x)=\sqrt{x^2+2x+5}-2\sin(x)-x^2+3$, $x^*=2.3319676558839640$.

$f_6(x)=\sin(x)e^x-2x-5$, $x^*=-2.5232452307325549$.

$f_7(x)=x^3-10$, $x^*=2.1544346900318837$.

$f_8(x)=e^{x^2+7x-30}-1$, $x^*=3.0000000000000000$.

$f_9(x)=x^5+x-10000$, $x^*=6.3087771299726891$.

$f_{10}(x)=\sqrt{x}-\frac{1}{x}-3$, $x^*=9.6335955628326952$.

$f_{11}(x)=e^x+x-20$, $x^*=2.8424389537844471$.

$f_{12}(x)=\ln(x)+\sqrt{x}-5$, $x^*=8.3094326942315718$.

The computational results in Table 1 show that the method (5) requires less NFE than G2, G3, G4, G5, G6, G7 and G8. The method (5) has iteration stabilities to the original iteration value. Therefore, they are of practical interest and can compete with other methods.

4 CONCLUSION

Based on three-step iterative method [17], we give further modification of the method to obtain higher-order convergence iterative method. Several examples show that the new method presented in the paper is more efficient and performs better than classical Newton's method and some other methods.

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